

# A New Integrable Model of (1+1)-Dimensional Dilaton Gravity Coupled to Toda Matter

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## Abstract

A new class of integrable two-dimensional dilaton gravity theories, in which scalar matter fields satisfy the Toda equations, is proposed. The simplest case of the Toda system is considered in some detail, and on this example we outline how the general solution can be obtained. We also demonstrate how the wave-like solutions of the general Toda systems can be simply derived. In the dilaton gravity theory, these solutions describe nonlinear waves coupled to gravity. A special attention is paid to making the analytic structure of the solutions of the Toda equations as simple and transparent as possible, with the aim to apply the idea of the separation of variables to non-integrable theories.

## 1 Introduction

The theories of (1+1)-dimensional dilaton gravity coupled to scalar matter fields are known to be reliable models for some aspects of higher-dimensional black holes, cosmological models, and waves. The connection between higher and lower dimensions was demonstrated in different contexts of gravity and string theory and, in several cases, has allowed finding the general solution or special classes of solutions in high-dimensional theories<sup>1</sup>. A generic example is the spherically symmetric gravity coupled to Abelian gauge fields and massless scalar matter fields. It exactly reduces to a (1+1)-dimensional dilaton gravity and can be explicitly solved if the scalar fields are constants independent of the coordinates. These solutions can describe interesting physical objects – spherical static black holes and simplest cosmologies. However, when the scalar matter fields, which presumably play a significant cosmological role, are nontrivial, not many exact analytical solutions of high-dimensional theories are known<sup>2</sup>. Correspondingly, the two-dimensional models of dilaton gravity that nontrivially couple to scalar matter are usually not integrable.

To obtain integrable models of this sort one usually has to make serious approximations, in other words, to deform the original two-dimensional model obtained by direct dimensional reductions of realistic higher-dimensional theories<sup>3</sup>. Nevertheless, the deformed models can qualitatively describe

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<sup>1</sup>See, e.g., [1]-[27] for a more detailed discussion of this connection, references, and solution of some integrable two-dimensional and one-dimensional models of dilaton gravity.

<sup>2</sup>See, e.g., [8], [11], [12], [17]-[23]; a review and further references can be found in [25], [26] and [23]

<sup>3</sup>We note that several important four-dimensional space-times with symmetries defined by two commuting Killing vectors may also be described by two-dimensional models of dilaton gravity coupled to scalar matter. For example, cylindrical gravitational waves can be described by a (1+1)-dimensional dilaton gravity coupled to one scalar field [28]-[30], [22]. The stationary axially symmetric pure gravity ([31], [11]) is equivalent to a (0+2)-dimensional dilaton gravity coupled to one scalar field. Similar but more general dilaton gravity models were also obtained in string theory. Some of them can be solved by using modern mathematical methods developed in the soliton theory (see e.g. [1], [2], [11], [19]).

certain physically interesting solutions of higher-dimensional gravity or supergravity theories related to the low-energy limit of superstring theories.

In our previous work (see, e.g., [20] - [23] and references therein) we constructed and studied some explicitly integrable models based on the Liouville equation. Recently, we attempted to find solutions of some realistic two-dimensional dilaton gravity models (derived from higher-dimensional gravity theories by dimensional reduction) using a generalized separation of variables introduced in [21], [22]. These attempts showed that seemingly natural ansatzes for the structure of the separation, which proved a success in previously studied integrable models, do not give interesting enough solutions ('zero' approximation of a perturbation theory) in realistic nonintegrable models. Thus an investigation of more complex dilaton gravity models, which are based on the two dimensional Toda chains, was initiated. Here we briefly present some results of this investigation. In particular, we propose a simplest class of models, which can be explicitly solved in terms of the solutions of the Toda equations. These solutions as well as their moduli space will be presented in a very simple and convenient form that allows for a simple description of the analytic and asymptotic properties of the solutions. At the same time, this representation is extremely convenient for their reductions to the wave solutions that include static and cosmological ones. This construction essentially and naturally generalizes the previous results, [20] - [23], and shows that essentially more complex structures of the separation of variables should be employed in realistic theories of gravity.

## 2 General model of (1+1)-dimensional dilaton gravity minimally coupled to scalar matter fields.

The effective Lagrangian of the (1+1)-dimensional dilaton gravity coupled to scalar fields  $\psi_n$  obtainable by dimensional reductions of a higher-dimensional spherically symmetric (super)gravity can usually be (locally) transformed to the form (see [20] - [23] for a detailed motivation and specific examples):

$$\mathcal{L} = \sqrt{-g} \left[ \varphi R(g) + V(\varphi, \psi) + \sum_{mn} Z_{mn}(\varphi, \psi) g^{ij} \partial_i \psi_m \partial_j \psi_n \right]. \quad (1)$$

Here  $g_{ij}(x^0, x^1)$  is the (1+1)-dimensional metric with the signature (-1,1),  $g \equiv \det|g_{ij}|$ , and  $R$  is the Ricci curvature of the two-dimensional space-time,

$$ds^2 = g_{ij} dx^i dx^j, \quad i, j = 0, 1. \quad (2)$$

The effective potentials  $V$  and  $Z_{mn}$  depend on the dilaton  $\varphi(x^0, x^1)$  and on  $N - 2$  scalar fields  $\psi_n(x^0, x^1)$  (we note that the matrix  $Z_{mn}$  should be negative definite to exclude the so called 'phantom' fields). They may depend on other parameters characterizing the parent higher-dimensional theory (e.g., on charges introduced in solving the equations for the Abelian fields). Here we consider the 'minimal' kinetic terms with the diagonal and constant  $Z$ -potentials,  $Z_{mn}(\varphi, \psi) = \delta_{mn} Z_n$ . This approximation excludes the important class of the sigma - model - like scalar matter discussed, e.g., in [27]; such models can be integrable if  $V \equiv 0$  and  $Z_{mn}(\varphi, \psi)$  satisfy certain rather stringent conditions. In (1) we also used the Weyl transformation to eliminate the gradient term for the dilaton.

To simplify derivations, we write the equations of motion in the light-cone metric,

$$ds^2 = -4f(u, v) du dv.$$

By first varying the Lagrangian in generic coordinates and then passing to the light-cone coordinates we obtain the equations of motion ( $Z_n$  are constants!)

$$\partial_u \partial_v \varphi + f V(\varphi, \psi) = 0, \quad (3)$$

$$f\partial_i(\partial_i\varphi/f) = \sum Z_n(\partial_i\psi_n)^2, \quad i = u, v. \quad (4)$$

$$2Z_n\partial_u\partial_v\psi_n + fV_{\psi_n}(\varphi, \psi) = 0, \quad (5)$$

$$\partial_u\partial_v \ln |f| + fV_\varphi(\varphi, \psi) = 0, \quad (6)$$

where  $V_\varphi = \partial_\varphi V$ ,  $V_{\psi_n} = \partial_{\psi_n} V$ . These equations are not independent. Actually, (6) follows from (3) – (5). Alternatively, if (3), (4), and (6) are satisfied, one of the equations (5) is also satisfied.

The higher-dimensional origin of the Lagrangian (1) suggests that the potential is the sum of the exponentials of linear combinations of the scalar fields,  $q_n^{(0)}$ , and of the dilaton  $\varphi$ <sup>4</sup>. In our previous work [23] we studied the constrained Liouville model, in which the system of the equations of motion (3), (5) and (6) is equivalent to the system of the independent Liouville equations for the linear combinations of the fields  $q_n \equiv F + q_n^{(0)}$ , where  $F \equiv \ln |f|$ . The easily derived solutions of these equations should satisfy the constraints (4), which was the most difficult part of the problem. The solution of the whole problem revealed an interesting structure of the moduli space of the solutions that allowed us to easily identify static, cosmological and wave-like solutions and effectively embed these essentially one-dimensional (in some broad sense) solutions into the set of all two-dimensional solutions and study their analytic and asymptotic properties.

Here we propose a natural generalization of the Liouville model to the model in which the fields are described by the Toda equations (or by nonintegrable deformations of them). To demonstrate that the model shares many properties with the Liouville one and to simplify a transition from the integrable models to nonintegrable theories we suggest a different representation of the Toda solutions, which is not directly related to their group - theoretical background.

Consider the theory defined by the Lagrangian (1) with the potential:

$$V = \sum_{n=1}^N 2g_n \exp q_n^{(0)}, \quad Z_n = -1, \quad (7)$$

where

$$q_n^{(0)} \equiv a_n\varphi + \sum_{m=3}^N \psi_m a_{mn}. \quad (8)$$

In what follows we also use

$$q_n \equiv F + q_n^{(0)} \equiv \sum_{m=1}^N \psi_m a_{mn}, \quad (9)$$

where  $\psi_1 + \psi_2 \equiv \ln |f| \equiv F$  ( $f \equiv \varepsilon e^F$ ,  $\varepsilon = \pm 1$ ),  $\psi_1 - \psi_2 \equiv \varphi$  and hence  $a_{1n} = 1 + a_n$ ,  $a_{2n} = 1 - a_n$ .

Rewriting the equations of motion in terms of  $\psi_n$ , we find that Eqs. (3) - (6) are equivalent to  $N$  equations of motion for  $N$  functions  $\psi_n$ ,

$$\partial_u\partial_v\psi_n = \varepsilon \sum_{m=1}^N \epsilon_n a_{nm} g_m e^{q_m}; \quad \epsilon_1 = -1, \quad \epsilon_n = +1, \text{ if } n \geq 2, \quad (10)$$

and two constraints,

$$\partial_i^2 \varphi \equiv \partial_i^2 (\psi_1 - \psi_2) = - \sum_{n=1}^N \epsilon_n (\partial_i \psi_n)^2, \quad i = u, v. \quad (11)$$

With arbitrary parameters  $a_{mn}$ , these equations of motion are not integrable. But as proposed in [16] - [18], [20] [23], Eqs.(10) are integrable and constraints (11) can be solved if the  $N$ -component vectors  $v_n \equiv (a_{mn})$  are pseudo-orthogonal.

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<sup>4</sup>Actually, the potential  $V$  usually contains terms non exponentially depending on  $\varphi$  (e.g., linear in  $\varphi$ ), and then the exponentiation of  $\varphi$  is only an approximation, see the discussion in [23].

Now, consider more general nondegenerate matrices  $a_{mn}$  and define the new scalar fields  $x_n$ :

$$x_n \equiv \sum_{m=1}^N a_{nm}^{-1} \epsilon_m \psi_m, \quad \psi_n \equiv \sum_{m=1}^N \epsilon_n a_{nm} x_m. \quad (12)$$

In terms of these fields, Eqs.(10) read as

$$\partial_u \partial_v x_m \equiv \varepsilon g_m \exp \sum_{k,n=1}^N \epsilon_n a_{nm} a_{nk} x_k \equiv \exp \sum_{k=1}^N A_{mk} x_k, \quad (13)$$

and we see that the symmetric matrix

$$A \equiv a^T \epsilon a, \quad \epsilon_{mn} \equiv \epsilon_m \delta_{mn}, \quad (14)$$

defines the main properties of the model.

If  $A$  is a diagonal matrix, we return to the  $N$ -Liouville model. If  $A$  is the Cartan matrix of a Lie algebra, the system (13) coincides with the corresponding Toda system, which is integrable and can be more or less explicitly solved (see, e.g., [32], [33]).<sup>5</sup> Here we mostly consider the  $A_N$  Toda systems having very simple solutions. However, the solutions have to satisfy the constraints that in terms of  $x_n$  are:

$$2 \sum_{n=1}^N a_n \partial_i^2 x_n = - \sum_{n,m=1}^N \partial_i x_m A_{mn} \partial_i x_n, \quad i = u, v. \quad (15)$$

In the  $N$ -Liouville model the most difficult problem was to solve the constraints (15) but this problem was eventually solved. In the general nonintegrable case of an arbitrary matrix  $A$  we do not know even how to approach this problem. We hope that in the Toda case the solution can be somehow derived but this problem is not addressed here. Instead, in Section 4 we introduce a simplified model that can be completely solved.

Now, let us write the general equations in the form that is particularly useful for the Toda systems. Introducing notation

$$X_n \equiv \exp(-\frac{1}{2} A_{nn} x_n), \quad \Delta_2(X) \equiv X \partial_u \partial_v X - \partial_u X \partial_v X, \quad \alpha_{mn} \equiv -2A_{mn}/A_{nn}, \quad (16)$$

it is easy to rewrite Eqs.(13) in the form:

$$\Delta_2(X_n) = -\frac{1}{2} \varepsilon g_n A_{nn} \prod_{m \neq n} X_m^{\alpha_{nm}}. \quad (17)$$

The multiplier  $|\varepsilon g_n A_{nn}|$  can be removed by using the transformation  $x_n \mapsto x_n + \delta_n$  and the final (standard) form of the equations of motion is

$$\Delta_2(X_n) = \varepsilon_n \prod_{m \neq n} X_m^{\alpha_{nm}}, \quad (18)$$

where  $\varepsilon_n \equiv \pm 1$ .

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<sup>5</sup> It can easily be seen that, due to the special structure of  $a_{mn}$  ( $a_{1n} = 1 + a_n$ ,  $a_{2n} = 1 - a_n$ ), the Cartan matrices of the simple algebras of rank 2 and 3 cannot be represented in the form (14). Further analysis shows that this probably is also true for any rank. As will be shown in a forthcoming publication, any symmetric matrix  $A_{mn}$ , which is the direct sum of a diagonal  $L \times L$ -matrix  $\gamma_n^{-1} \delta_{mn}$  and of an arbitrary symmetric matrix  $\bar{A}_{mn}$ , can be represented in form (14). If  $\bar{A}_{mn}$  is a Cartan matrix, the system (13) reduces to  $L$  independent Liouville (Toda  $A_1$ ) equations and the higher-rank Toda system.

These equations are in general not integrable. However, when  $A_{mn}$  are the Cartan matrices, they simplify to integrable equations (see [32]). For example, for the Cartan matrix of  $A_N$ , only the near-diagonal elements of the matrix  $\alpha_{mn}$  are nonvanishing,  $\alpha_{n-1,n+1} = 1$ . This allows one to solve Eq.(18) for any  $N$ . The parameters  $\alpha_{mn}$  are invariant w.r.t. transformations  $x_n \mapsto \lambda_n x_n + \delta_n$  and hence  $A_{mn}$  can be made non-symmetric while preserving the standard form of the equations (recall that the Cartan matrices of  $B_N$ ,  $C_N$ ,  $G_2$ , and  $F_4$  are not symmetric). In this sense,  $\alpha_{mn}$  are the fundamental parameters of the equations of motion. From this point of view, the characteristic property of the Cartan matrices is the simplicity of Eqs.(18) which allows one to solve them by a generalization of separation of variables. As is well known, when  $A_{mn}$  is the Cartan matrix of any simple algebra, this procedure gives the exact general solution (see [32]). In next Section we show how to construct the exact general solution for the  $A_N$  Toda system and write a convenient representation for the general solution that differs from the standard one given in [32].

### 3 Solution of the $A_N$ Toda system

The  $A_N$  equations are extremely simple,

$$\Delta_2(X_n) = \varepsilon_n X_{n-1} X_{n+1}, \quad X_0 \mapsto 1, \quad X_{N+1} \mapsto 1, \quad n = 1, \dots, N, \quad (19)$$

and can be reduced to one equation for  $X_1$  by using the relation between  $\Delta_2(X)$  and higher determinants,  $\Delta_n(X)$  (see [32]):

$$\Delta_2(\Delta_n(X)) = \Delta_{n-1}(X) \Delta_{n+1}(X), \quad \Delta_1(X) \equiv X, \quad n \geq 2. \quad (20)$$

From Eqs.(19), (20) we find that

$$\Delta_{N+1}(X_1) = \prod_n \varepsilon_n. \quad (21)$$

This equation looks horrible but is known to be soluble.

Let us start with the Liouville ( $A_1$  Toda) equation  $\Delta_2(X) = g$  (see [34], [35], [32], [23]). Calculating the derivatives of  $\Delta_2(X)$  w.r.t.  $u$  and  $v$ , we find that

$$\partial_u^2(X^{-1} \partial_v^2 X) = 0, \quad \partial_v(X^{-1} \partial_u^2 X) = 0. \quad (22)$$

It follows that if  $X$  satisfies (22) then there exist some ‘potentials’  $\mathcal{U}(u)$ ,  $\mathcal{V}(v)$  such that

$$\partial_u^2 X - \mathcal{U}(u) X = 0, \quad \partial_v^2 X - \mathcal{V}(v) X = 0. \quad (23)$$

Thus the Liouville solution can be written as ([23])

$$X(u, v) = \sum a_\mu(u) C_{\mu\nu} b_\nu(v), \quad (24)$$

where  $a_\mu(u)$  and  $b_\nu(v)$  ( $\mu, \nu = 1, 2$ ) are linearly independent solutions of the equations

$$a''(u) - \mathcal{U}(u) a(u) = 0, \quad b''(v) - \mathcal{V}(v) b(v) = 0. \quad (25)$$

and  $C_{\mu\nu}$  is a nonsingular matrix. As the potentials are unknown, the solutions  $a_1, b_1$  can be taken arbitrary while  $a_2, b_2$  then may be defined by the Wronskian first-order equations

$$W[a_1(u), a_2(u)] = 1, \quad W[b_1(v), b_2(v)] = 1. \quad (26)$$

The matrix  $C_{\mu\nu}$  should obviously satisfy the normalization condition  $\det C = g$ .

We have repeated this well known derivation at some length because it is completely applicable to the  $A_N$  Toda equation (21). By similar but rather cumbersome derivations it can be shown that  $X_1$  satisfy the equations

$$\partial_u^{N+1} X + \sum_{n=0}^{N-1} \mathcal{U}_n(u) \partial_u^n X = 0, \quad \partial_v^{N+1} X + \sum_{n=0}^{N-1} \mathcal{V}_n(v) \partial_v^n X = 0. \quad (27)$$

Thus the solution of (21) can be written in the same ‘separated’ form (24), where now  $a_\mu(u)$  and  $b_\nu(v)$  satisfy the ordinary linear differential equations of the order  $N+1$  (corresponding to Eqs.(27)), with the unit Wronskians,

$$W[a_1(u), \dots, a_{N+1}(u)] = 1, \quad W[b_1(v), \dots, b_{N+1}(v)] = 1, \quad (28)$$

and  $\det C = \prod \varepsilon_n$ .

As an exercise, we suggest the reader to prove these statements for  $N = 2$ . The key relation that follows from the condition  $\partial_u \Delta_3(X) = 0$  is the partial integral

$$\partial_v \left[ \partial_v \left( \frac{X}{\partial_u X} \right) / \partial_v \left( \frac{\partial_u^3 X}{\partial_u X} \right) \right] = 0. \quad (29)$$

It follows that the expression in the square brackets is equal to an arbitrary function  $A_0(u)$  and thus we have

$$\partial_v \left[ \left( \frac{X}{\partial_u X} \right) + A_0(u) \left( \frac{\partial_u^3 X}{\partial_u X} \right) \right] = 0. \quad (30)$$

Denoting the expression in the square bracket by  $-A_1(u)$  and introducing the notation  $\mathcal{U}_1(u) = A_1/A_0$  and  $\mathcal{U}_0(u) = A_1^{-1}$ , we get Eq.(27) with  $N = 2$ .

Let us return to the general solution of Eq.(21). In fact, considering Eqs.(28) as inhomogeneous differential equations for  $a_{N+1}(u)$ ,  $b_{N+1}(v)$  with arbitrary chosen functions  $a_n(u)$ ,  $b_n(v)$  ( $1 \leq n \leq N$ ), it is easy to write the explicit solution of this problem:

$$a_{N+1}(u) = \sum_{n=1}^N a_n(u) \int^u d\bar{u} W_N^{-2}(\bar{u}) M_{N,n}(\bar{u}). \quad (31)$$

Here  $W_N$  is the Wronskian of the arbitrary chosen functions  $a_n$  and  $M_{N,n}$  are the complementary minors of the last row in the Wronskian. Replacing  $a$  by  $b$  and  $u$  by  $v$  we can find the expression for  $b_{N+1}(v)$  from the same formula (31). To complete the solution we should derive the expressions for all  $X_n$  in terms of  $a_n$  and  $b_n$ . This can be done with simple combinatorics that allows one to express  $X_n$  in terms of the  $n$ -th order minors. For example, it is very easy to derive the expressions for  $X_2$ :

$$X_2 = \varepsilon_1 \Delta_2(X_1) = \varepsilon_1 \sum_{i < j} W[a_i(u), a_j(u)] W[b_i(v), b_j(v)],$$

which is valid for any  $N \geq 1$  ( $i, j = 1, \dots, N+1$ ). Note that expressions for all  $X_n$  have a similar separated form. This possibly hints that some rather complex separation of variables may give us a tool for (approximate) solving more general, nonintegrable equations (18).

Our simple representation of the  $A_N$  Toda solution is completely equivalent to what one can find in [32] but is more convenient for treating some problems. For example, it is useful in discussing asymptotic and analytic properties of the solutions of the original physics problems. It is especially appropriate for constructing wave-like solutions of the Toda system which is similar to the wave solutions of the  $N$ -Liouville model. In fact, quite like the Liouville model, the Toda equations support the wave-like solutions. To derive them let us first identify the moduli space of the Toda

solutions. Recalling the  $N$ -Liouville case, we may try to identify the moduli space with the space of the potentials  $\mathcal{U}_n(u)$ ,  $\mathcal{V}_n(v)$ . Possibly, this is not the best choice and, in fact, in the Liouville case we finally made a more cumbersome choice suggested by the solution of the constraints. For our present purposes the choice of the potentials is as good as any other because each choice of  $\mathcal{U}_n(u)$  and  $\mathcal{V}_n(v)$  defines some solution and, vice versa, any solution given by the set of the functions  $(a_1(u), \dots, a_{N+1}(u))$ ,  $(b_1(v), \dots, b_{N+1}(v))$  satisfying the Wronskian constraints (28) defines the corresponding set of the potentials  $(\mathcal{U}_0(u), \dots, \mathcal{U}_{N-1}(u))$ ,  $(\mathcal{V}_0(v), \dots, \mathcal{V}_{N-1}(v))$ .

Now, as in the Liouville case, we may consider the reduction of the moduli space to the space of constant ‘vectors’  $(U_0, \dots, U_{N-1})$ ,  $(V_0, \dots, V_{N-1})$ . The fundamental solutions of the equations (27) with these potentials are exponentials (in the nondegenerate case)

$$a_n(u) = \exp(\mu_n u), \quad b_n(v) = \exp(\nu_n v), \quad \sum_{n=1}^{N+1} \mu_n = 0, \quad \sum_{n=1}^{N+1} \nu_n = 0. \quad (32)$$

In this reduced case we may regard the space of the parameters  $(\mu_n, \nu_n)$  the new moduli space, in complete agreement with the Liouville case. Of course, the constraints (11) define further restrictions on the moduli  $(\mu_n, \nu_n)$  but here we do not address this problem.

Note only that, as in the  $N$ -Liouville case, one can construct nonsingular waves. To show this is not much more difficult than in the Liouville case but requires more lengthy derivations. We hope to publish these in a forthcoming paper.

## 4 A simple integrable model of (1+1)-dimensional dilaton gravity coupled to Toda scalar matter

Let us suppose that the potential  $V$  is independent of  $\varphi$ , i.e.  $V(\varphi, \psi) \equiv V(\psi)^6$ . Then Eq.(6) is simply the D’Alembert equation for  $F(u, v)$ . It follows that the metric can be written as

$$f = \varepsilon a'(u) b'(v).$$

Due to the residual freedom of the coordinate choice in the light-cone metric we can choose  $(a, b)$  as the new (local) coordinates and then denote them by  $(u, v)$ . In this coordinates and notation we simply have  $f = \varepsilon$  and  $F = 0$  in all the equations. We thus see that the equations (5) are independent of  $\varphi$  and can be solved independently of the equations (3), (4). Suppose that the matter fields  $\psi$  are known and first solve equations (3) and (4).

The general solution of Eq.(3) can be written as

$$\varphi = -\varepsilon \int_0^u \int_0^v d\bar{u} d\bar{v} V[\psi(\bar{u}, \bar{v})] + A(u) + B(v), \quad (33)$$

where  $A(u)$ ,  $B(v)$  are arbitrary functions. The constraints (4) in this model have the form

$$\partial_i^2 \varphi = - \sum_{n=1}^N (\partial_i \psi_n)^2, \quad i = u, v. \quad (34)$$

Using (5) we easily derive

$$\partial_i V = \varepsilon \partial_j \sum_1^N (\partial_i \psi_n)^2, \quad (i, j) = (u, v) \text{ or } (v, u), \quad (35)$$

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<sup>6</sup>In this model we suppose that there are  $N$  scalar matter fields  $\psi_n$  with  $n = 1, \dots, N$  while  $F$  is trivial and  $\varphi$  is treated separately

find  $A(a)$ ,  $B(b)$  in terms  $\psi$ , and finally obtain:

$$\varphi = -\varepsilon \int_0^u \int_0^v d\bar{u} d\bar{v} V[\psi(\bar{u}, \bar{v})] - \int_0^u d\bar{u} \int_0^{\bar{u}} d\bar{u} \Phi_u(\bar{u}) - \int_0^v d\bar{v} \int_0^{\bar{v}} d\bar{v} \Phi_v(\bar{v}) + A'(0)u + B'(0)v, \quad (36)$$

where we omitted the unimportant arbitrary term  $A(0) + B(0) = \varphi(0, 0)$  and denoted

$$\Phi_u(u) \equiv \sum_1^N (\partial_u \psi_n(u, 0))^2, \quad \Phi_v(v) \equiv \sum_1^N (\partial_v \psi_n(v, 0))^2.$$

Now, to get integrable equations for  $\psi$  we take the potential (7) with  $q_n^{(0)}$  given by the r.h.s. of Eq.(9). Then, we can use for the scalar fields the equations (10) and (12) – (14). If we take the potential for which the  $\psi$  equations of motion can be reduced to integrable Toda equations we find an explicit solution for the nontrivial class of dilaton gravity minimally coupled to scalar matter fields. This model is a very complex generalization of the well studied CGHS model in which the scalar fields are free and  $V = g$ . In future, we plan a detailed study of the  $A_N$  case. The easiest case is  $N = 1$  (the Liouville equation for one  $\psi$ ). The first really interesting but simple theory is the case of two scalar fields satisfying the  $A_2$  Toda equations. Taking, for example,

$$V = \exp(\sqrt{3} \psi_1 - \psi_2) + \exp(2 \psi_2),$$

we find the simplest realization of the  $A_2$  Toda dilaton gravity model the complete solution of which can be obtained by use of the above derivations.

As a simple exercise one may consider the reduction from dimension (1+1) both to the dimension (1+0) (‘cosmological’ reduction) and to the dimension (0+1) (‘static’ or ‘black hole’ reduction) as well as the moduli space reduction to waves. One of the most interesting problems for future investigations is the connection between these three objects. It was discovered in the  $N$ -Liouville theory but now we see that it can be found in a much more complex theory described by the Toda equations. It is not impossible that the connection also exists (in a weaker form?) in some nonintegrable theories, say, in theories close to the Toda models.

Note in conclusion, that the one-dimensional Toda equations were earlier employed mostly in connection with the cosmological and black hole solutions (see, e.g. [36] – [38]). To include into consideration the waves one has to step up at least on dimension higher. The principal aim of the present paper was to make the first step and explore this problem in a simplest two-dimensional Toda environment.

A more detailed presentation will be published elsewhere.

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